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THREE TYPES OF WAVELETS AND APPLICATIONS

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ABSTRACT: This paper discusses algorithms adapted for a construction of three types of wavelets. We build in the rst case scaling functions and associated wavelets which are fast decreasing. In the second case, we construct wavelets which are exponential decreasing. In the last one, we construct scaling functions which are regular and have compact support and we prove that the associated wavelets have compact support and preserve the original regularity of the scaling functions. As applications, we prove that the wavelet bases constructed in this paper are adapted for the study of regular functional spaces $C^s(\mathbb{R})$ and $H^s(\mathbb{R})$ $(s \in \mathbb{R})$ and are easy to implement.

Keywords: Scaling function, Wavelet, Riesz basis, Dual basis, Multiresolution analysis, Sobolev space.

Mathematics Subject Classication : 42C15, 44A15.

1. Introduction

Wavelets are functions generated from one basis function by dilations and translations. Wavelet concepts have unfolded their full computational efficiency mainly in harmonic analysis (for the study of Calderon-Zygmund operators) and in signal analysis. The wavelet expansions induce isomorphisms between function and sequence spaces. It means that certain Sobolev or Besov norms of functions are equivalent to weighted sequence norms for the coecients in their wavelet expansions. The wavelets have cancellation properties that are usually expressed in terms of vanishing polynomial moments. The combination of the two previous properties of wavelets provides a rigorous analysis of adaptative schemes for elliptic problems. Moreover, nonlinear approximation is an important concept related to adaptative approximation. The standard technique in physical geodesy is based on an expansion of the investigated functions which are in general homogeneous harmonic polynomials. This technique has obvious disadvantages because the most essential drawback is the global support of the basis. By using wavelets with compact support, we obtain a highly localized resolution by increasing the maximum degree of the truncated singular value decomposition to extreme sizes. Moreover, they are applied to numerical problems.

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We construct in this paper three types of wavelets. We build in the first case scaling functions and associated wavelets which are fast decreasing. In the second case, we construct wavelets which are exponential decreasing. In the last one, we construct scaling functions which are regular and have compact support. There is a relation between regularity and the support. The scaling function is constructed in an elementary way. The main contribution offered in this paper which diers from the other constructions is the realization of global higher regularity by more elementary techniques than perhaps those involved in (Ajmi et Al (2001), Dahmen (2000) and Jouini (2007)). The global regularity is sucient for applications and the bases are easy to implement. The direct method used in this paper constitutes a very important method for the study of many problems of mathematics and physics because we give a good description of scaling functions and associated wavelets specially in the case of compact support. Wavelets described in this work have many applications as computation and numerical simulation for elliptic problems or image processing.

Section 2 is devoted to the construction of three types of scaling functions which will be useful for the remainder of the work.

In section 3, we describe an algorithm of construction of the three types of associated wavelets. These constructions are not complicated and not technical because the scaling functions are constructed in an elementary way. In particular, general wavelet bases with higher regularity and compact support are given in this topic.

In the last Section, to apply algorithms, we characterize regular spaces namely Sobolev spaces in terms of discrete norm equivalences.

2. Construction of the Scaling Function

We begin with the notion of multiresolution analysis introduced by Mallat (1989) and associated properties that are essential in the proofs of our results.

Definition 2.1

A multiresolution analysis is a family $(V_j)_{j\in\mathbb{Z}}$ of closed linear subspaces of $L^2(\mathbb{R})$ such that:

i) $V_j \subset V_{j+1}$; for every $j \in \mathbb{Z}$.

ii) $f(x) \in V_j$ if only if $f(2x) \in V_{j+1}$.

iii) $\bigcap_{j\in\mathbb{R}} V_{j\in\mathbb{Z}} = \{0\}$ and $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$:

iv) $f(x) \in V_0$ if only if $f(x-k) \in V_0$; for every $k \in \mathbb{Z}$

v) There exists a function $g(x)$ in V_0 such that the system of functions $(g(x - k))_{k\in\mathbb{Z}}$ forms a Riesz basis of V_0.

Let P_j be the orthogonal projector from $L^2(\mathbb{R})$ onto V_j. Then, we have

$$\lim_{j\to-\infty}\|P_jf\|_2 = 0, \quad \lim_{j\to+\infty}\|f - P_jf\|_2 = 0,$$

Definition 2.2

i) A multiresolution analysis is regular if there exists a constant C such that for every integer m and for every $x \in \mathbb{R}$, we have the following property:

$$|g(x)| \leq C(1 + |x|)^{-m} \tag{2.1}$$

ii) A multiresolution analysis is r–regular, if the scaling function $g(x)$ and its first r derivations satisfy the property (i).

We have the following result.

Lemma 2.1 Let $(g(x - k))_{k\in\mathbb{Z}}$ be a Riesz basis of V_0 such that the function g is r-regular. Let φ be the function dened by

$$\hat{\varphi}(\xi)(\Sigma_{k\in\mathbb{Z}}|\hat{g}(\xi + 2k\pi)|^2)^{-\frac{1}{2}} \tag{2.2}$$

then the system of functions $\varphi(x - k)$; $k \in \mathbb{Z}$; form an orthonormal basis of V_o and the scaling function φ is r-regular.

Proof. We must prove that the scaling function φ is r-regular because the orthogonality is immediate. The function g is r-regular, the function $\Sigma_{k\in\mathbb{Z}}|\hat{g}(\xi + 2k\pi)|^2$ belongs to $L^2([0, 2\pi])$ and is 2π–periodic. Then, $\varphi \in V_o$. It is clear that we have

$$\Sigma_{k\in\mathbb{Z}}|\hat{\varphi}(\xi + 2k\pi)|^2 = 1.$$

We first prove that if we write

$$F(\xi) = \sum_{\ell\in\mathbb{Z}} a_\ell e^{i\ell\xi},$$

then a_ℓ is fast decaying if and only if F is of class C^∞: In fact, we have

$$a_\ell \frac{1}{2\pi}\int_0^{2\pi} F(\xi)e^{-i\ell\xi}d\xi$$

and if $\ell \neq 0$, by using integration, we get, for all k,

$$|a_\ell| = \frac{1}{2\pi}\left|\frac{(-)^k}{\ell^k}\int_0^{2\pi} F^{(k)}(\xi)e^{-i\ell\xi}d\xi\right| \leq \frac{\sup_{\xi\in[0,2\pi]}|F^{(k)}(\xi)|}{\ell^k}.$$

Then, if F is of class C^∞, we get that a_ℓ is fast decaying. On the other hand, if a_ℓ is fast decaying, then the function $\Sigma_{\ell\in\mathbb{Z}} a_\ell\, e^{i\ell\xi}$ and all its derivatives are uniformly convergent and then F is of class C^∞. We denote

$$\sum_{k\in\mathbb{Z}} \left|\hat{g}(\xi+2k\pi)\right|^2 = \sum_{\ell\in\mathbb{Z}} a_\ell e^{i\ell\xi}. \tag{2.3}$$

Then, we have

$$a_\ell = \frac{1}{2\pi}\int_0^{2\pi} \sum_{k\in\mathbb{Z}} \left|\hat{g}(\xi+2k\pi)\right|^2 e^{-i\ell\xi} d\xi$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \left|\hat{g}(\xi)\right|^2 e^{-i\ell\xi} d\xi$$

$$= \int_{\mathbb{R}} g(x)\overline{g}(x+\ell)dx$$

If g is fast decaying, then the coecient a‘ is fast decaying. We deduce that the series $\sum_{k\in\mathbb{Z}} \left|\hat{g}(\xi+2k\pi)\right|^2$ is C^∞ and $\left(\sum_{k\in\mathbb{Z}} \left|\hat{g}(\xi+2k\pi)\right|^2\right)^{-1/2}$ is C^∞. If we write $\left(\sum_{k\in\mathbb{Z}} \left|\hat{g}(\xi+2k\pi)\right|^2\right)^{-1/2} = \sum_{\substack{\ell\in\mathbb{Z}\\ \ell}} b_\ell e^{i\ell\xi}$ where $(b_\ell)_\ell$ are coefficients of Fourier of $\left(\sum_{k\in\mathbb{Z}} \left|\hat{g}(\xi+2k\pi)\right|^2\right)^{-1/2}$, then $(b_\ell)_\ell$ are fast decaying and φ is also fast decaying. We get

$$\varphi(x) = \sum_{\ell\in\mathbb{Z}} b_\ell g(x-\ell). \tag{2.4}$$

We verify immediately that the derivatives of φ are also fast decaying; then we have

$$\varphi^{(k)}(x) = \sum_{\ell\in\mathbb{Z}} b_\ell g(x-\ell).$$

Lemma 2.1 is completely proved.

We recall that a function g is exponentially decaying if there exist two positive constants A and B such that for every $x \in \mathbb{R}$, we have

$$|g(x) \le A \exp(-B \mid x \mid)$$

We prove now the following result.

Proposition 2.1 We assume that the function g is exponentially decaying. Then, there exist two positive constants C and D such that

$$|\varphi(x)| \le C \exp(-D \mid x \mid).$$

We first prove the following technical lemma.

Lemma 2.2 Let $m(\xi) = \Sigma_{k\in\mathbb{Z}} a_k\, e^{ik\xi}$. The two properties are equivalent:

i) m is analytic in the band $|\, Im\xi \,| < A$.

ii) There exist two positive constants B and C such that for every $k \in \mathbb{Z}$, we have $a_k \leq C\exp(-B \mid k \mid)$ for all $B < A$.

Proof.

It is clear that the function m is still 2π-periodic because if $m(\xi + 2\pi) - m(\xi)$ vanishes on $[0, 2\pi]$, it vanishes on all the band and we have for $k > 0$:

$$a_k = \frac{1}{2\pi}\int_o^{2\pi} m(\xi)e^{-ik\xi}$$

$$= \frac{1}{2\pi}\left(\int_o^{-iB} m(\xi)e^{-ik\xi}d\xi + \int_{-iB}^{-iB+2\pi} m(\xi)e^{-ik\xi}d\xi + \int_{-iB+2\pi}^{2\pi} m(\xi)e^{-ik\xi}d\xi\right).$$

$m(\xi)$ is 2π-periodic, the first and the third term vanished. We estimate the second term by the following expression:

$$2\pi \sup_{\xi\in[-iB,\, iB+2\pi]} \left|m(\xi)\right| e^{-kB}.$$

We have the same result for $k < 0$ by replacing $-iB$ by iB. The second implication is immediate. In fact, if g is exponential decaying, the series $\sum_{k\notin\mathbb{Z}} \left|\hat{g}(\xi + 2k\pi)\right|^2$ is analytic on the band. This function is strictly positive on $[0, 2\pi]$, the inverse function is still analytic on a band around the real axis.

Proof of Proposition 2.1. By using Lemma 2.2, we deduce that the coefficients b_ℓ defined on 2.4 are exponentially decaying, and then φ is also exponentially decaying. Proposition 2.1 is completely proved.

Lemma 2.3 We assume that the functions $(\varphi(x - k))_{k\in Z}$ form an orthonormal basis of V_0 and $f \in V_0$. Then, there exists a function $m(\xi)$ satisfying

$$\hat{f}(\xi) = m(\xi)\hat{\varphi}(\xi)$$

and such that $m(\xi)$ is 2π–periodic and belongs to $L^2([0; 2\pi])$.

Proof. We start from the equality

$$f(x) = \Sigma_{k\in\mathbb{Z}\alpha k\varphi}(x - k)$$

then

$$\hat{f}(\xi) = (\Sigma_{k\in\mathbb{Z}\alpha kc}{}^{-ik\xi})\hat{\varphi}(\xi)$$

Let

$$m(\xi) = \Sigma_{k\in\mathbb{Z}\alpha kc}{}^{-ik\xi}$$

then we obtain

$$\hat{f}(\xi) = m(\xi)\hat{\varphi}(\xi)$$

It is clear that $m(\xi)$ is 2π–periodic. By using the fact that the functions $(\varphi(x - k))_{k\in\mathbb{Z}}$ form an orthonormal basis of V_0 and the Plancherel formula, we obtain

$$\|f\|_2 = (\Sigma_{k\in\mathbb{Z}}|\in_k|^2)^{1/2}$$

$$= \frac{1}{\sqrt{2\pi}}\left(\int_{-\infty}^{+\infty} |m(\xi)|^2 |\hat{\varphi}(\xi)|^2 \, d\xi\right)^{1/2}$$

$$= \frac{1}{\sqrt{2\pi}}\left(\sum_{k\in\mathbb{Z}}\int_{2k\pi}^{(2k+1)x} |m(\xi)|^2 |\hat{\varphi}(\xi)|^2 \, d\xi\right)^{1/2}$$

$$= \frac{1}{\sqrt{2\pi}}\left(\sum_{k\in\mathbb{Z}}\int_{0}^{2\pi} |m(\xi)|^2 |\hat{\varphi}(\xi + 2k\pi)|^2 \, d\xi\right)^{1/2}$$

$$= \frac{1}{\sqrt{2\pi}}\left(\int_{0}^{2\pi} |m(\xi)|^2 \, d\xi\right)^{1/2}$$

because $\sum_{k\in\mathbb{Z}}|\hat{\varphi}\xi + 2k\pi|^2 = 1$. Then $m(\xi)$ belongs to $L^2([0, 2\pi])$. Lemma 2.3 is proved.

Remark 2.1 In wavelet theory, the function $m(\xi)$ is called filter.

We study now the relation between the filter $m(\xi)$ and the support of the scaling function.

Proposition 2.2 We assume that the functions $(\varphi(x - k))_{k\in\mathbb{Z}}$ form an orthonormal basis of V_0 and $m_o(\xi) = \Sigma_{k\in\mathbb{Z}\alpha k}e^{-ik\xi}$ the filter satisfying $\hat{\varphi}(2\xi) = m_o(\xi)\varphi(\xi)$. Then

i) If φ has a compact support, then the coefficients α_k are equal to zero except a finite number of indexes k.

ii) In particular $m_o(\xi)$ is a trigonometric polynomial.

Proof. Since $V_{-1} \subset V_0$, we write

$$\frac{1}{2}\varphi\left(\frac{x}{2}\right) = \sum_{k\in\mathbb{Z}}\alpha_k\varphi(x - k),$$

where

$$\alpha_k = \frac{1}{2}\int_{-\infty}^{+\infty}\left(\varphi\left(\frac{x}{2}\right) - \varphi(x+k)\right)dx.$$

Then

$$\hat{\varphi}(2\xi) = \left(\sum_{k\in\mathbb{Z}} \alpha_k e^{-ik\xi}\right)\hat{\varphi}(\xi),$$

and

$$m_o(\xi) = \sum_{k\in\mathbb{Z}} \alpha_k e^{-ik\xi}.$$

If φ has a compact support in $[-A;\ A]$, then the expression of α_k becomes

$$\alpha_k = \frac{1}{2}\int_{-A-k}^{A-k}\left(\varphi\left(\frac{x}{2}\right)0 - \varphi(x+k)\right)dx$$

We verify that for $|k| > 3A$; we have $\alpha_k = 0$ because $\sup\varphi\left(\frac{\pi}{2}\right) \subset [-2A,$ $2A]$. We have then the property (i) of Proposition 2.2. (ii) is immediate.

Proposition 2.3 Let $m_o(\xi) = \sum_{k=N_1}^{Nl2} \alpha_k e^{-ik\xi}$ be a trigonometric polynomial satisfying $m_o(0) = 1$. Then the product $\Pi_{j=1}^{\infty} m_o(2^{-j}\xi)$ converges simply on $\mathbb{R}$. Moreover, the convergence is uniform on every compact.

Proof. We have

$$|m_o(\xi) - 1| = \left|\sum_{k=N_1}^{N_2} \alpha_k e^{-ik\xi} - 1\right|$$

$$= \left|\sum_{k=N_1}^{N_2} \alpha_k e^{-ik\xi} - \sum_{k=N_1}^{N_2} \alpha_k\right|$$

$$= \left|\sum_{k=N_1}^{N_2} \alpha_k (e^{-ik\xi} - 1)\right|$$

$$\leq \sum_{k=N_1}^{N_2} |\alpha_k|\left|e^{-ik\xi} - 1\right|$$

$$\leq 2\sum_{k=N_1}^{N_2} |\alpha_k|\left|\sin\left(\frac{k\xi}{2}\right)\right|$$

then

$$\leq 2\sum_{k=N_1}^{N_2} |\alpha_k| \left|\sin\left(\frac{k\xi}{2}\right)\right|$$

Since $\log(|m_o(2^{-j}\xi)|)$ is equivalent at infinity to $|m_o(2^{-j}\xi) - 1|$, it is sufficient then to prove that $\sum_{j=1}^{\infty} \left|m_0(2^{-j}\xi) - 1\right|$ converges simply on $\mathbb{R}$ and uniformly on every compact. By using the equality described above, we have

$$|m_o(2^{-j}\xi) - 1| \leq C2^{-j}|\xi|$$

and

$$\sum_{j=1}^{\infty} 2^{-j} - \infty$$

then, $\sum_{j=1}^{\infty} \left|m_0(2^{-j}\xi) - 1\right|$ converges simply on $\mathbb{R}$. We assume now that $|\xi| \leq B$, we obtain

$$\sup_{|\xi|\leq B} |m_o(2^{-j}\xi) - 1| \leq CB2^{-j},$$

then

$$\sum_{j=1}^{\infty} \sup_{|\xi|\leq B} \left|m_o(2^{-j}\xi) - 1\right| < \infty$$

We deduce that $\sum_{j=1}^{\infty} \left|m_o(2^{-j}\xi) - 1\right|$ converges uniformly on $[-B; B]$ and then on every compact.

We have precisely the following result which characterizes the scaling function defined by the infinite product.

Proposition 2.4 Let $m_o(\xi) = \sum_{k=N_1}^{N_2} \alpha_k e^{-ik\xi}$ be a trigonometric polynomial function such that $m_o(0) = 1$ and $|m_o(\xi)|^2 + |m_o(\xi + \pi)|^2 = 1$. Then the infinite product $\Pi_{j=1}^{\infty} m_o(2^{-j}\xi)$ is an entire function of exponential type.

Remark 2.2 The innite product $\Pi_{j=1}^{\infty} m_o(2^{-j}\xi)$ is an entire function of exponential type and is the Fourier transform of a distribution with compact support in $[N_1; N2]$:

We need in the remainder of this work the following known lemma.

Lemma 2.4 (Lemma of F. Riesz) Let $A(\xi)$ a positive trigonometric polynomial invariant by the transformation $\xi \to -\xi$. Then, there exists a trigonometric polynomial $B(\xi)$ of degree $M\left(B(\xi)=\sum_{m=0}^{M} b_m e^{im\xi},\, b_m \in \mathbb{R}\right)$ such that $|B(\xi)|^2 = A(\xi)$:

Corollary 2.1 There exists a trigonometric polynomial $m_o(\xi)$ satisfying the following properties:

$$m_o(0) = 1$$
$$|m_o(\xi)|^2 + |m_o(\xi+\pi)|^2 = 1;$$
$$m_o(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N L(\xi),$$

where $L(\xi)$ is a trigonometric polynomial and N an non-null integer.

Remark 2.3 The scaling function φ defined by

$$\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_o(2^{-j}\xi),$$

belongs to $L^2(\mathbb{R})$. The last point is to realize orthogonality of the sequence $(\varphi(x-k))_{k\in\mathbb{Z}}$.

Proposition 2.5 Let ~~L~~mo be a trigonometric polynomial satisfying the following properties:

$$m_o(0) = 1$$
$$|m_o(\xi)|^2 + |m_o(\xi+\pi)|^2 = 1;$$
$$m_o(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^N L(\xi),$$

Let φ be the function of $L^2(\mathbb{R})$ defined by:

$$\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_o(2^{-j}\xi).$$

If we denote $B = \sup_{k\in\mathbb{Z}}|L(\xi)|$, then, we have:

$$\left|\hat{\varphi}(\xi)\right| \leq C(1+|\xi|)^{-N+\frac{\log B}{\log 2}}.$$

Proof. There exists a constant c such that

$$\left|\hat{\varphi}(\xi)\right| \leq 1 + c|\xi| \leq \exp^{c|\xi|}$$

then

$$\left|\hat{\varphi}(\xi)\right| \leq \prod_{j=1}^{\infty} \exp(2^{-j}c|\xi|)$$

$$\leq \exp\left(c|\xi| \sum_{j=1}^{\infty} 2^{-j} \right)$$

$$\leq \exp(c|\xi|).$$

For $|\xi| \leq 1$; there exists a constant C' such that

$$|\hat{\varphi}(\xi)| \leq C'$$

then

$$|\hat{\varphi}(\xi)| \leq C'(1+|\xi|)^{-N+\log B/\log 2}.$$

For $|\xi| > 1$; we have

$$|m_o(2^{-j}\xi)| = \left| \frac{1+e^{i2^{-j}\xi}}{2} \right|^N L(2^{-j}\xi),$$

or

$$\left|1+e^{i2^{-j}\xi}\right| = 2\left|\cos(2^{-j-1}\xi\right|$$

then

$$\left| \frac{1+e^{i2^{-j}\xi}}{2} \right|^N \left|\cos(2^{-j-1}\xi)\right|^N$$

On the other hand, we have

$$\prod_{j=1}^{\infty} \left| \frac{1+e^{i2^{-j}\xi}}{2} \right|^N = \prod_{j=1}^{\infty} \left| \frac{1+e^{i2^{-j}\xi}}{2} \right|^N$$

$$= \left(\prod_{j=1}^{\infty} \left|\cos(2^{-j-1}\xi)\right| \right)^N$$

$$= \left| \prod_{j=1}^{\infty} \cos(2^{-j-1}\xi) \right|^N$$

$$= \left| \frac{\sin \frac{\xi}{2}}{\frac{\xi}{2}} \right|^N$$

then

$$|\hat{\varphi}(\xi)| = \left|\frac{\sin\frac{\xi}{2}}{\frac{\xi}{2}}\right|^N \left|\prod_{j=1}^{\infty} L(2^{-j}\xi)\right|$$

We deduce that for $|\xi| > 1$; there exists $j_o \in N$ such that

$$2^{j_o-1} \leq |\xi| < 2^{j_o}$$

or again

$$\frac{\log|\xi|}{\log 2} < j_o \leq 1 + \frac{\log|\xi|}{\log 2}.$$

Then

$$\left|\prod_{j=1}^{\infty} L(2^{-j}\xi)\right| = \left|\prod_{j=1}^{j_o} L(2^{-j}\xi)\right| \left|\prod_{j=j_o+1}^{\infty} L(2^{-j}\xi)\right|$$

Since

$$\prod_{j=j_o+1}^{\infty} L(2^{-j}\xi) = \prod_{j=1}^{\infty} L(2^{-j-j_o}\xi)$$

$$= \prod_{j=1}^{\infty} L(2^{-j}(2^{-j-j_o}\xi))$$

therefore, for $|2^{-j_o}\xi| < 1$, there exists a constant C such that

$$\left|\prod_{j=1}^{\infty} L(2^{-j}(2^{-j_o}\xi))\right| \leq cte$$

and

$$\left|\prod_{j=1}^{\infty} L(2^{-j}\xi)\right| \leq B^{j_o} = \exp(j_o \log B)$$

$$\leq \exp\left(\log B + \log B \frac{\log|\xi|}{\log 2}\right)$$

$$\leq \exp\left(\log|\xi|^{\frac{\log B}{\log 2}}\right) = B|\xi|^{\frac{\log B}{\log 2}}$$

$$\leq B(1 + |\xi|)^{\frac{\log B}{\log 2}}$$

then

$$\left|\prod_{j=1}^{\infty} L(2^{-j}\xi)\right| \leq cte(1+|\xi|)^{\frac{\log B}{\log 2}}.$$

Or

$$\left|\frac{\sin\frac{\xi}{2}}{\frac{\xi}{2}}\right| \leq \frac{cte}{1+|\xi|}$$

therefore

$$\left|\frac{\sin\frac{\xi}{2}}{\frac{\xi}{2}}\right| \leq cte(1+|\xi|)^{-N}$$

and then

$$|\hat{\varphi}(\xi)| \leq C(1+|\xi|)^{-N+\frac{\log B}{\log 2}}$$

Proposition 2.5 is proved.

Proposition 2.6 We assume that mo satisfies the hypotheses of Proposition 2.5. Then we have

$$\forall n \in \mathbb{N}, |\widehat{u_n}(\xi)| \leq C(1+|\xi|)^{-N+\frac{\log B}{\log 2}}$$

where C is a constant independent of n.

Proof. We have

$$|\widehat{u_n}(\xi)| = \prod_{j=1}^{n} |m_o(2^{-j}\xi)| \chi_{[-\pi,\pi]}(2^{-n}\xi)$$

$$= \prod_{j=1}^{n} \left|\frac{1+e^{i2^{-j}\xi}}{2}\right|^N |L(2^{-j}\xi)| \chi_{[-\pi,\pi]}(2^{-n}\xi)$$

$$= \left[\prod_{j=1}^{n} \left|\frac{1+e^{i2^{-j}\xi}}{2}\right|\right]^N \prod_{j=1}^{n} |L(2^{-j}\xi)| \chi_{[-\pi,\pi]}(2^{-n}\xi)$$

then

$$\left|\widehat{u_n}(\xi)\right| = \left[\prod_{j=1}^{n} \cos(2^{-j-1}\xi)\right]^N \prod_{j=1}^{n} \left|L(2^{-j}\xi)\right| \chi_{[-\pi,\pi]}(2^{-n}\xi).$$

Since

$$\prod_{j=1}^{n} \cos(2^{-j-1}\xi) = \prod_{j=2}^{n+1} \cos(2^{-j-1}\xi) = \frac{\sin\left(\frac{\xi}{2}\right)}{2^n \sin(2^{-n-1}\xi)}.$$

For $-\pi \leq 2^{-n}\xi \leq \pi$, we have

$$\left|2^{-n-1}\xi\right| \leq \frac{\pi}{2}$$

then

$$\left|\sin(2^{-n-1}\xi)\right| \geq \frac{2}{\pi} 2^{-n-1} |\xi|$$

therefore

$$\frac{1}{\left|\sin(2^{-n-1}\xi)\right|} \chi_{[-\pi,\pi]}(2^{-n}\xi) \leq \frac{2^n \pi}{|\xi|}$$

and

$$\frac{\sin\left(\frac{\xi}{2}\right)}{\left|2\sin(2^{-n-1}\xi)\right|} \chi_{[-\pi,\pi]}(2^{-n}\xi) \leq \frac{2}{\pi} \left|\frac{\sin\frac{\xi}{2}}{\frac{\xi}{2}}\right|$$

$$\leq cte(1 + |\xi|)^{-1}.$$

To prove the Proposition 2.6, it is sufficient to prove that

$$\prod_{j=1}^{n} \left|L(2^{-j}\xi)\right| \chi_{[-\pi,\pi]}(2^{-n}\xi) \leq C(1 + |\xi|)^{\frac{\log B}{\log 2}}$$

We have $L(0) = 1$: Then, by using Lemma 2.2, we get

$$|L(\xi)| \leq 1 + c|\leq| \leq \exp(c|\xi|)$$

then

$$\prod_{j=1}^{n} \left|L(2^{-j}\xi)\right| \leq \prod_{j=1}^{n} \exp(c2^{-j}|\xi|)$$

$$\leq \exp\left(\sum_{j=1}^{n} 2^{-j}\,|\xi|\right)$$

$$\leq \exp\left(c\sum_{j=1}^{\infty} 2^{-j}2^{-j}\,|\xi|\right)$$

$$\leq \exp(c|\xi|)$$

Then, for $|\xi| \leq \pi$, the product $\Pi_{j=1}^{n} 2^{-j}\left|L(2^{-j}\xi)\right|$ is uniformly bounded. We deduce that there exists $c > 0$ such that

$$\prod_{j=1}^{n}\left|L(2^{-j}\xi)\right| \leq c$$

and then, for $|\xi| \leq \pi$, we have

$$\prod_{j=1}^{n}\left|L(2^{-j}\xi)\right| \leq c(1+|\xi|)^{\frac{\log B}{\log 2}}$$

We study now the case $|\xi| \geq \pi$. There exists $j_o \in \mathbb{N}$ such that

$$2^{j_o}\pi \leq |\xi| < 2^{j_o+1}\pi.$$

We remark two cases.

i) If $j_o > n$; we have

$$2^{j_o}\pi > 2^{n}\pi$$

then

$$|\xi| > 2^{n}\pi$$

and therefore

$$\chi_{[-\pi,\pi]}(2^{-n}\xi) = 0$$

and the needed property is satisfied.

ii) If $j_o \leq n$, we have

$$\prod_{j=1}^{n}\left|L(2^{-j}\xi)\right| = \prod_{j=1}^{j_o}\left|L(2^{-j}\xi)\right| \prod_{j=j_o+1}^{n}\left|L(2^{-j}\xi)\right|$$

then

$$\prod_{j=j_o+1}^{n}\left|L(2^{-j}\xi)\right| = \prod_{j=j_o+1}^{n}\exp(c2^{-j}\,|\xi|)$$

$$\leq \exp\left(c \sum_{j=j_o+1}^{n} 2^{-j} |\xi| \right)$$

we obtain

$$\prod_{j=j_o+1}^{n} \left| L(2^{-j}\xi) \right| \leq \exp\left(\sum_{k=0}^{\infty} 2^{-k} \left| 2^{-(j_o+1)} \xi \right| \right)$$

$$\leq \exp(2c|2^{-(j_o+1)}\xi|)$$

$$\leq \exp(c|\xi|).$$

On the other hand, we have

$$\prod_{j=1}^{j_o} \left| L(2^{-j}\xi) \right| \leq B^{j_o}$$

then, for $2^{j_o}\pi \leq |\xi|$, we have

$$\log \pi + j_o \log 2 \leq \log |\xi|$$

so

$$j_0 \leq \frac{\log|\xi|}{\log 2} - \frac{\log \pi}{\log 2}$$

and then

$$B^{j_0} = \exp(j_0 \log B) \leq \exp\left[\left(\frac{\log|\xi|}{\log 2} - \frac{\log \pi}{\log 2} \right) \log B \right]$$

$$\leq \exp\left(\frac{\log|\xi| \log B}{\log 2} \right) \exp\left(-\frac{\log \pi \log B}{\log 2} \right)$$

$$\leq cte \exp\left(\log |\xi|^{\frac{\log B}{\log 2}} \right)$$

$$\leq cte |\xi|^{\frac{\log B}{\log 2}}$$

$$\leq cte(1 + |\xi|)^{\frac{\log B}{\log 2}}$$

We get

$$\forall |\xi| \geq \pi, \prod_{j=1}^{\infty} \left| L(2^{-j}\xi) \right| \chi_{[-\pi,\pi]}(2^{-n}\xi) \leq cte(1 + |\xi|)^{\frac{\log B}{\log 2}}.$$

Proposition 2.6 is proved.

Proposition 2.7 We use hypotheses of Proposition 2.6 and we assume that $B < 2^{N-\frac{1}{2}}$. Then, $\hat{u}_n$ converges to $\hat{\varphi}$ in the sense of $L^2(\mathbb{R})$:

Proof. We have $\hat{u}_n$ converges simply to $\hat{\varphi}$. Then, $|\hat{u}_n - \hat{\varphi}|^2$ tends to 0: Using Propositions 2.5 and 2.6, we have

$$B < 2^{N-\frac{1}{2}} \Rightarrow \log B < \left(N - \frac{1}{2}\right)\log 2$$

$$\Rightarrow -2N + 2\frac{\log B}{\log 2} < -1$$

$$\Rightarrow (1 + |\xi|)^{-2N+\frac{2\log B}{\log 2}} \in L^1(\mathbb{R}).$$

Dominated convergence theorem gives that $\int_{\mathbb{R}} |\widehat{u_n}(\xi) - \hat{\varphi}(\xi)|^2 \, d\xi$ tends to 0.

3. Construction of the associated wavelet

For $j \in \mathbb{Z}$, we denote by W_j the orthogonal complement of V_j in V_{j+1}. It is clear that we have

i) $F \in W_j$ if only if $F\left(\frac{\cdot}{2^j}\right) \in W_0$.

ii) $F \in W_j$ if only if $F\left(\cdot - \frac{\cdot}{2^j}\right) \in W_j$.

iii) The spaces W_j are orthogonal and we have $\oplus_j W_j = L^2(\mathbb{R})$.

Proposition 3.1 We assume that the function φ is fast decaying. Then, there exists a function ψ such that

i) The functions $(\psi(x-k))_{k\in\mathbb{Z}}$ form an orthonormal basis of W_o.

ii) The functions $(2^{j/2}\psi(2^jx-k))_{k\in\mathbb{Z}}$ form an orthonormal basis of W_j.

iii) The functions $(2^{j/2}\psi(2^jx-k))_{j,k\in\mathbb{Z}}$ form an orthonormal basis of $L_2(\mathbb{R})$.

Proof. The function φ belongs to V_0. Then, we have

$$\varphi\left(\frac{\pi}{2}\right) \in V_{-1} \subset V_0.$$

Then there exists a sequence $(\alpha_k)_{k\in\mathbb{Z}}$ such that

$$\frac{1}{2}\varphi\left(\frac{\pi}{2}\right) = \Sigma_{k\in\mathbb{Z}}\alpha_k\varphi(x+k)$$

where

$$\alpha_k = \frac{1}{2}\int_{\mathbb{R}}\varphi\left(\frac{\pi}{2}\right)\overline{\varphi}(x+k)dx$$

The function φ is fast decaying, then the sequence $(\alpha_k)_{k\in\mathbb{Z}}$ is fast decaying. By using Fourier transformations, we obtain (3.1)

$$\hat{\varphi}(2\xi) = m_o(\xi)\hat{\varphi}(\xi) \tag{3.1}$$

where

$$m_o(\xi) = \Sigma_{k\in\mathbb{Z}}\alpha_k e^{ik\xi}$$

We have

$$\Sigma_{k\in\mathbb{Z}}\left|\hat{\varphi}(\xi+2k\pi)\right|^2 = 1,$$

then

$$\Sigma_{k\in\mathbb{Z}}\left|\hat{\varphi}(2a\xi+2k\pi)\right|^2 = 1,$$

it gives

$$\sum_{k\in\mathbb{Z}}\left|m_o(\xi+k\pi)\right|^2\left|\hat{\varphi}(\xi+k\pi)\right|^2 = 1$$

We divide the series in two parts, we get

$$\sum_{k\in\mathbb{Z}}\left|m_o(\xi)\right|^2\left|\hat{\varphi}(\xi+2k\pi)\right|^2 + \sum_{k\in\mathbb{Z}}\left|m_o(\xi+\pi)\right|^2\left|\hat{\varphi}(\xi+2k\pi)\right|^2 = 1,$$

it gives

$$|m_o(\xi)|^2 + |m_o(\xi+\pi)|^2 = 1. \tag{3.2}$$

The functions $\left(\frac{1}{\sqrt{2}}\hat{\varphi}\left(\frac{x}{2}-k\right)\right)_{k\in\mathbb{Z}}$ form an orthonormal basis of V_{-1}. Then we have

$$\widehat{V}_{-1} = \{m(2\xi)\hat{\varphi}(2\xi),\ m \text{ is } 2\pi - \text{periodic and locally in } L^2(\mathbb{R})\}$$

$$= \{m(2\xi)m_o(\xi)\hat{\varphi}(\xi),\ m \text{ is } 2\pi - \text{periodic and locally in } L^2(\mathbb{R})\}$$

Then

$$\widehat{V}_{-1} = \{m(2\xi)\hat{\varphi}(2\xi),\ m \text{ is } 2\pi - \text{periodic and locally in } L^2(\mathbb{R})\}$$

and the application $F \to m$ is an isometry from V_o onto $L^2([0.2\pi])$.

To construct a basis of the space $\widehat{W}_{-1}$, we look for a filter $\ell(\xi)$ which is 2π periodic and orthogonal to functions $m(2\xi)m_o(\xi)$ where m is a function 2π-periodic. This condition is expressed as

$$\int_o^{2\pi} m(2\xi)m_o(\xi)\overline{\ell(\xi)}d\xi = 0.$$

or

$$\int_o^{\pi} m(2\xi)(m_o(\xi)\overline{\ell(\xi)}d\xi + m_o(\xi+\pi)\overline{\ell(\xi+\pi)}d\xi = 0.$$

We get

$$m_o(\xi)\overline{\ell(\xi)}d\xi + m_o(\xi+\pi)\overline{\ell(\xi+\pi)}d$$

The last condition is satisfied if there exists a function $\theta(\xi)$ such that

$$\forall\xi \in [0,\pi], \begin{Bmatrix} \ell(\xi) & = 0(\xi)\overline{m_0(\xi+\pi)}; \\ \ell(\xi+\pi) & = -\theta(\xi)\overline{m_0(\xi)}. \end{Bmatrix}$$

and then, if $\theta(\xi) = e^{-i\xi}$, we must find a π-periodic function μ such that

$$\ell(\xi) = \mu(\xi)e^{-i\xi}\overline{m_0(\xi+\pi)}.$$

We take as a basis of $\widehat{W}_{-1}$ the functions

$$\sqrt{2}e^{-i\xi}\overline{m_0(\xi+\pi)}\hat{\varphi}(\xi)e^{2ik\xi},\ k \in \mathbb{Z}$$

By dilation, it gives a choice for the function ψ

$$\hat{\psi}(\xi) = e^{-i\xi/2}\overline{m_0\left(\frac{\xi}{2}+\pi\right)}\hat{\varphi}(\xi/2) \tag{3.3}$$

Or

$$\psi(x) = \sum_{k\in\mathbb{Z}}(-1)^k\overline{\alpha_{1-k}}\varphi(2x-k).$$

Remark 3.1 If φ is fast decaying, ψ has the same property and if φ is exponential decaying, φ is also exponential decaying.

Proposition 3.2 If the functions $(2^{j/2}\psi(2^jx-k))_{j,k\in\mathbb{Z}}$ form an orthonormal basis of $L^2(\mathbb{R})$ and ψ and the r derivatives are fast decaying, we have

$$\forall \ell \leq, \int_{\mathbb{R}} x^{\ell} \psi(x) dx = 0.$$

Proof. We use the recurrence method on r. For $r = 1$, we have $\ell = 0$. But

$$\int_{\mathbb{R}} \psi(x) dx = \hat{\psi}(0).$$

Or $\left|\hat{\varphi}(0)\right| = 1$; then the property (3.3) gives

$$\hat{\psi}(0) = \overline{m_0(\pi)}$$

On the other hand, we have

$$\hat{\varphi}(\xi) = \prod_{j \geq 0} m_o\left(\frac{\xi}{2^j}\right)$$

then, $|m_o(0)| = 1$ and the property (3.2) gives $m_o(\pi) = 0$. We assume that the result is true for all $\ell \leq r - 1$. Let $x_o = k_o 2^{-J}$ such that $\psi(r)(x_o) \neq 0$. Let $j \geq J$ and $k = 2^j x_o$. The orthogonality of the wavelets and the Taylor-Young formula give:

$$0 = \int_{\mathbb{R}} \psi(x) 2^{j/2} \psi(2^j x - k) dx$$

$$= \int_{\mathbb{R}} \left(\sum_{\ell=0}^{r} \frac{(x - x_o)}{\ell\,!} \psi^{(\ell)}(x_o) + (x - x_o)^{r+1} O(1) \right) 2^{j/2} \psi(2^j x - k) dx$$

The $O(1)$ is uniform on $\mathbb{R}$. We divide the integral into two terms and by using a change of variables $u = 2j(x - x_o)$, we obtain

$$0 = \int_{\mathbb{R}} \frac{(x - x_o)^r}{r\,!} \psi^{(r)}(x_o) 2^{j/2} \psi(2^j x - x_o)) dx$$

$$+ \int_{\mathbb{R}} (x - x_o)^{r+1} O(1) 2^{j/2} \psi(2^j x - x_o)) dx$$

$$= \frac{\psi^{(r)}(x_o)}{r\,!} 2^{-\left(\frac{1}{2}+r\right)j} \int_{\mathbb{R}} u^r \psi(u) du + O\left(2^{-\left(\frac{1}{2}+r+1\right)j}\right)$$

$$= \frac{\psi^{(r)}(x_o)}{r\,!} \int_{\mathbb{R}} u^r \psi(u) du + O(2^{-j})$$

For a big j, we deduce

$$\int_{\mathbb{R}} u^r \psi(u) du = 0.$$

Proposition 3.2 is completely proved.

Remark 3.2

i) This property means that the function $\hat{\psi}$ has zero of order equal or bigger than $(r-1)$ at 0.

ii) The wavelets on multidimensional case can be constructed from the wavelets on dimension one by using the tensor product (Jouini (1993); Jouini et Al (1992)) and then they satisfy the same properties described above.

We describe the algorithm of construction of orthogonal multire solution analysis with compact support. We denote

$$P_N(y) = \sum_{k=0}^{N-1} C_{N-1+k}^k y^k,\ N \in \mathbb{N}^*.$$

We choose an odd polynomial R and we define the polynomial P by

$$P_N(y) = P_N(y) + y^N R\left(\frac{1}{2} - y\right)$$

with the properties:

$$\forall y = P_N(y) + y^N R\left(\frac{1}{2} - y\right)$$

and

$$\sup_{0 \le v \le 1}\left[P_N(y) + y^N R\left(\frac{1}{2} - y\right)\right] = \sup_{0 \le v \le 1} P(y) < 2^{2N-1}$$

We denote

$$L(\xi) = P\left(\sin^2 \frac{\xi}{2}\right).$$

L is a positive polynomial. The Riesz lemma proves that there exists a polynomial ℓ such that $|\ell(\xi)|^2 = L(\xi)$ and $\ell(0) = 1$.

We take now

$$m_o(\xi) = \left(\frac{1 + e^{-i\xi}}{2}\right)^N \ell(\xi).$$

Then, mo satisfies the conditions of Proposition 2.5 and we have:

$$\sup_{\xi \in \mathbb{R}} |\ell(\xi)| = \left[\sup_{\xi \in \mathbb{R}} |\ell(\xi)|^2\right]^{\frac{1}{2}}$$

$$\left(\sup_{\xi\in\mathbb{R}} L(\xi)\right)^{\frac{1}{2}} = \left(\sup_{\xi\in\mathbb{R}} P\left(\sin^2\frac{\xi}{2}\right)\right)^{\frac{1}{2}}$$

$$= \left(\sup_{0\le y\le 1} P(y)\right)^{\frac{1}{2}}$$

$$\le (2^{2N-1})^{\frac{1}{2}} = 2^{N-\frac{1}{2}}.$$

We obtain an orthogonal multiresolution analysis and an orthonormal wavelet basis with compact support.

Remark 3.3 To construct the orthogonal multiresolution analysis of *I*. Daubechies (1988), we use the method described in the previous section with a particular choice of a null polynomial R. Then, we have:

$$P(y) = P_N(y),$$

$$L(\xi) = P_N\left(\sin^2\frac{\xi}{2}\right),$$

$$\sup_{\xi\in\mathbb{R}} P_N(y) < 2^{2N-1},$$

and

$$\sup_{\xi\in\mathbb{R}} \left|L_N(\xi)\right| < 2^{N-\frac{1}{2}}.$$

4. The Study of Regular Spaces

We prove in this section that the wavelet bases constructed in this paper are adapted for the study of the spaces $C^s(\mathbb{R})$ and $H^{-s}(\mathbb{R})$.

Definition 4.1

i) A function f belongs to $C^s_{x_0}$ if there exists a polynomial P of degree lower or equal to entire party of s such that

$$f(x) = P(x - x_o) + O(|x - x_o|^s).$$

ii) $f \in C^s(\mathbb{R})$ if $f \in C^s_{x_0}$ for every $x_o \in \mathbb{R}$ and if $O(x)$ is uniform in x_o.

Theorem 4.1 We assume that $s \in \mathbb{R}/\mathbb{N}$, the wavelet ψ is C^{s+1} and has zero first moments. If we denote

$$C_{j,k} = \int_{\mathbb{R}} f(x)\psi_{j,k}(x)dx,$$

then, we have the following equivalence

$$f \in C^s(\mathbb{R}) \text{ if only if } \left|C_{j,k}\right| \le C2^{-\left(\frac{1}{2}+s\right)_j}.$$

Proof. If $f \in C^s$, we have

$$\left|C_{j,k}\right| = \left|\int_{\mathbb{R}} f(x)\psi_{j,k}(x)dx,\right|$$

$$= \left|\int_{\mathbb{R}} \left(f(x) - P\left(x - \frac{k}{2^j}\right)\right)\psi_{j,k}(x)dx,\right|$$

(because ψ has some vanishing moments and is fast decaying) then

$$\left|c_{j,k}\right| \le C\int_{\mathbb{R}} \left|x - k2^{-j}\right|^s \frac{2^{\frac{1}{2}}dx}{\left(1 + 2^j\left|x - k2^{-j}\right|\right)^N}$$

$$\le C2^{-\left(\frac{1}{2}+s\right)j}\int_{\mathbb{R}} \frac{|y|^s\, dy}{(1+|y|)^N}$$

$$\le C2^{-\left(\frac{1}{2}+s\right)j}$$

Reciprocally, we assume that

$$\left|C_{j,k}\right| \le C2^{-\left(\frac{1}{2}+s\right)j}$$

We denote

$$Q_j(f) = \sum_k C_{j,k}\psi_{j,k}.$$

We have

$$\left\|Q_j(f)\right\|_\infty \le C2^{-s_j}$$

and

$$\left\|\partial^\alpha(Q_j(f)\right\|_\infty \le C2^{(\alpha-s)_j}$$

(due to the localization property of ψ).

Let $x_o \in \mathbb{R}$. We denote

$$P_j(x - x_o) = \sum_{j<\alpha} \frac{(x - x_o)^\alpha}{\alpha!} Q_j^{(\alpha)}(f)(x_o)$$

and

$$P_j(x - x_o) = \sum_{j \geq o} P_j(x - x_o).$$

This series converges. Then if j_o is such that

$$2^{-j_o} \leq |x - x_o| < 2^{-j_o+1}.$$

we have

$$\left|f(x) - P(x - x_o)\right| \leq \sum_{j \leq j_o} \left|Q_j(f)(x) - P_j(x - x_o)\right| + \sum_{j > j_o} \left|Q_j(f)(x) - P_j(x - x_o)\right|$$

The first sum is increased by

$$\sum_{j \leq j_o} |x - x_o|^{[s]+1} \sup_{|\alpha|=|s|+1} \left\|\partial^\alpha Q_j(f)\right\|_\infty \leq C \sum_{j \leq j_o} |x - x_o|^{[s+1]} 2^{([s]-s+1)_j}$$

The second sum is increased by

$$C \sum_{j > j_o} \left(\left\|Q_j(f)\right\|_\infty + \sum_{\alpha < |s|} |x - x_o|^\alpha \left\|Q_j^{(\alpha)}(f)\right\|_\infty \right)$$

$$\leq C \sum_{j > j_o} \left(2^{-sj} + \sum_{\alpha < |s|} |x - x_o|^\alpha 2^{-(j-\alpha)} \right)$$

$$\in C|x - x_o|^\alpha.$$

Proposition 4.1 If $f \in H^s(\mathbb{R})(s > 0)$, then we have the following inequality

$$||f - P_j(f)||_2 \leq C2^{-JS} ||f||_{H^s}.$$

Proof. If $f \in H^s(\mathbb{R})$; we have the inequality

$$\sum_{j,k} \left|C_{j,k}\right|^2 (1 + 2^{2j})^s \leq C \|f\|_{H^s}^2 .$$

Then

$$\|f - P_J(f)\|_2^2 = \sum_{j \geq J} \left|C_{j,k}\right|^2$$

$$\leq 2^{-2Js} \sum_{j \geq J} (1 + 2^{2j}) \left|C_{j,k)}\right|^2$$

$$\leq C2^{-2Js} \|f\|_{H^s}^2 .$$

Proposition 4.1 is then proved.

The characterization of $H^s(\mathbb{R})$ is immediate. We can now establish the following result.

Theorem 4.2 Assume that φ is a C^{s+1} -function, then we have

i) for $f \in L^2(\mathbb{R})$, $\|f\|_2 \approx \|P_0 f\|_2 + \left(\sum_{j\geq 0}\|Q_j f\|_2^2\right)^{\frac{1}{2}}$

ii) For $(s > 0)$, we have

$$f \in H^s(\mathbb{R}) \text{ if only if } P_0 f \in L^2(\mathbb{R}) \text{ and } \sum_{j\geq 0} 4^{js}\|Q_j f\|_2^2 < \infty.$$

5. Conclusion

In this paper, we described algorithms adapted for construction of general scaling function and associated wavelet. These functions are regular and have compact support. Then, we constructed multiresolution analyses. As applications, we proved that these analyses are well adapted for the study of the spaces $C^s(\mathbb{R})$ and $H^s(\mathbb{R})$.

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